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EXTENDED BOUNDING THEOREMS FOR NONLINEAR LIMIT ANALYSIS†

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Abstract—A stability criterion for rigid–perfectly plastic structural analysis for structures undergoing large deformation is formulated, which plays a key role in nonlinear limit analysis. Based on the property of the superpotential in this theory, various variational techniques are suggested for optimal bounds of the plastic limit load factor. An application is illustrated.

1. INTRODUCTION

The classical limit analysis theory neglects changes in geometry of structures (see Prager and Hodge, 1951), and shows a beautiful symmetry between the upper and lower bounding theorems for safety factors by using abstract description (Gao, 1988a,b). Since the influence of changes in geometry on the yield load of a structure is of great significance in both theoretical problems and engineering applications, many efforts in this direction have been made by Onat and Haythornthwaite (1956), Onat (1960) and Sawczuk (1964) etc. Their results showed that geometry changes can work for or against the capacity of the strength of the structure. However, the basic features of the mathematical problems of determining the limit load factor in the finite deformation plasticity, remained somewhat obscure.

Recently, the extremum properties of variational functions were studied by Gao and Strang (1989a) for nonlinear boundary value problems. By introducing a so-called complementary gap function, a remarkable symmetry between the primal and dual variational problems was discovered. Application to nonlinear limit analysis yields a pair of dual bounding theorems, when the gap has the right sign (Gao and Strang, 1990). The plastic yield condition in this theory is relaxed by the complementary plastic superpotential. In the present paper, we will find that this gap function plays a key role in nonlinear limit analysis. Based on the property of the complementary superpotential, various variational principles are constructed. An application of this is illustrated by an example of rigid–perfectly plastic beam analysis.

2. COMPLEMENTARY BOUNDING THEOREMS

Let us consider in the reference configuration, a rigid–perfectly plastic body occupying an open, bounded, connected region $\Omega \subset \mathbb{R}$, with piecewise smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_u \cup \Gamma_t$, $\Gamma_u \cap \Gamma_t = \emptyset$. On Γ_t, the surface traction is given as $\bar{t} = v_c t$, where $v_c > 0$ is the load multiplier and t is the unit load distribution. Let $\mathcal{U}$ and $\mathcal{S}$ be the admissible displacement space and Kirchhoff stress space, respectively. In the case of large displacements but small strains, the boundary value problem for limit analysis is finding the plastic load multiplier v_c and field functions $(u,S) \in \mathcal{U} \times \mathcal{S}$ such that

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$$\left.\begin{array}{ll} \varepsilon_{ij}-\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})=0 & \text{in } \Omega \\ [(\delta_{ik}+u_{i,k})S_{kj}]_{,j}-b_i=0 & \text{in } \Omega \\ \varepsilon_{ij}\in\partial W^*(S_{ij}) & \text{in } \Omega \\ u_i=0 & \text{on } \Gamma_u \\ [(\delta_{ik}+u_{i,k})S_{kj}]n_j-v_c t=0 & \text{on } \Gamma_t \end{array}\right\} \tag{1}$$

in which W^* is the complementary plastic superpotential (Gao, 1988a,b),

$$W^*(S)=\begin{cases}0 & \text{if } S\in\mathcal{K}\\ + & \infty \text{ otherwise,}\end{cases} \tag{2}$$

where $\mathcal{K}$ is a convex subset of the admissible stress space $\mathcal{S}$:

$$\mathcal{K}=\{S\in\mathcal{S}\mid f(S)\leqslant 0 \quad \text{in } \Omega\}, \tag{3}$$

$f(S)$ is the plastic yield function, which is assumed to be convex and lower semi-continuous. According to the theory of convex analysis (cf. e.g. Ekland and Temam, 1976), $W^*:\mathcal{S}\to\mathbb{R}\cup\{+\infty\}$ is convex and lower semi-continuous. $\partial W^*(S)$ denotes the subdifferential of W^* at S, which is a convex subset of the strain space $\mathcal{E}$ (cf. e.g. Gao 1988a,b; Gao and Strang, 1989):

$$\partial W^*(S)=\begin{cases}\lambda\,\partial f(S)/\partial S_{ij} & \text{if } f(S)=0,\ \lambda\geqslant 0 \quad \text{in } \Omega\\ \{0\} & \text{if } f(S)<0 \quad \text{in } \Omega\\ \varnothing & \text{if } f(S)>0 \quad \text{in } \Omega,\end{cases} \tag{4}$$

where $\lambda\geqslant 0$ is the plastic flow factor.

The plastic superpotential W can be obtained by using the Legendre–Fenchel transformation (see Gao, 1988a; Gao and Strang, 1989b):

$$W(\varepsilon)=\sup_{S\in\mathcal{S}}\{S_{ij}\varepsilon_{ij}-W^*(S)\}=\sup_{S\in\mathcal{K}}S_{ij}\varepsilon_{ij}, \tag{5}$$

i.e. $W(\varepsilon)$ is the support function of the convex set $\mathcal{K}$, which is also convex and lower semi-continuous. So the inverse form of the constitutive relation (1.3) can be written as

$$S_{ij}\in\partial W(\varepsilon_{ij}). \tag{6}$$

Lemma. *The structure governed by eqn* (1) *is stable if*

$$G(u,S)=\int_\Omega \frac{1}{2}S_{ij}u_{k,i}u_{k,j}\,\mathrm{d}\Omega\geqslant 0. \tag{7}$$

Here $G:\mathcal{U}\times\mathcal{S}\to\mathbb{R}$ is the so-called complementary gap function of nonlinear variational boundary value problems (see Gao and Strang, 1989). It plays an important role in geometrical nonlinear mechanics (see Gao *et al.*, 1989–1990).

Let $\mathcal{U}_a\subset\mathcal{U}$ be the kinematically admissible space:

$$\mathcal{U}_a=\left\{v\in\mathcal{U}\mid v=0 \quad \text{on } \Gamma_u,\ \int_\Gamma tv\,\mathrm{d}\Gamma>0\right\}. \tag{8}$$

The kinematically admissible load factor $v_u:\mathcal{U}_a\to\mathbb{R}$ may be given as

$$v_u(v) = \frac{\int_\Omega W(\varepsilon(v))\, d\Omega - \int_\Omega b_i v_i \, d\Omega}{\int_{\Gamma_t} t_i v_i \, d\Gamma}. \tag{9}$$

One has the upper bound theorem for the limit load multiplier (Gao and Strang, 1990).

Theorem 1. *Among all $v \in \mathscr{U}_a$, the stationary points u of $v_u(v)$ and associated $S(u)$ solve the boundary value problem* (1). *If for a given $v \in \mathscr{U}_a$, the gap function G satisfies*

$$G(v, S(u)) = \int_\Omega \frac{1}{2} S_{ij}(u) v_{k,i} v_{k,j}\, d\Omega \geqslant 0, \tag{10}$$

then $v_u(v)$ gives an upper bound of the limit load multiplier v_c, i.e.

$$v_c \leqslant v_u(v). \tag{11}$$

If for any given $v \in \mathscr{U}_a$, inequality (10) *holds, then v_c minimizes v_u, i.e.*

$$v_c = \inf_{v \in \mathscr{U}_a} v_u(v). \tag{12}$$

According to the theory of convex analysis, the subdifferential constitutive relation $S(u) \in \partial W(\varepsilon(u))$ yields the following variational inequality:

$$\int_\Omega W(\varepsilon(v))\, d\Omega - \int_\Omega W(\varepsilon(u))\, d\Omega \geqslant \int_\Omega S_{ij}(u)\,[\varepsilon_{ij}(v) - \varepsilon_{ij}(u)]\, d\Omega \quad \forall v \in \mathscr{U}. \tag{13}$$

Substituting geometrical equations into (13) and using the Gauss–Green law, Theorem 1 can be proved (see Gao and Strang, 1989b).

On the dual side, let $\mathscr{S}_a \subset \mathscr{S} \times \mathscr{U}$ be the statically admissible space:

$$\mathscr{S}_a = \{(T, v) \in \mathscr{S} \times \mathscr{U} \mid [(\delta_{ik} + u_{i,k}) T_{kj}]_{,j} - b_i = 0 \quad \text{in } \Omega$$
$$(\delta_{ik} + u_{i,k}) T_{kj} n_j - v^- t_i = 0 \quad \text{on } \Gamma_t\}, \tag{14}$$

in which $v^- > 0$ is a statically admissible load multiplier associated with $(T, v) \in \mathscr{S}_a$. One puts

$$v_\ell(T, v) = v^-(T, v) - \int_\Omega W^*(T)\, d\Omega. \tag{15}$$

Then the lower bound theorem for limit load multiplier can be given by

Theorem 2. *Let (S, u) solve the boundary value problem* (1). *If for a given $(T, v) \in \mathscr{S}_a$, the gap function G satisfies*

$$G(u, S) - G(v, T) + G(u - v, T) \geqslant 0, \tag{16}$$

then $v_\ell(T, v)$ gives a lower bound for the limit load multiplier v_c:

$$v_c \geqslant v_\ell(T, v). \tag{17}$$

The proof of this theorem can be found in Gao and Strang (1989b). If the statically admissible field satisfies the plastic yield condition, i.e. $T \in \mathscr{K}$, then $W^*(T) = 0$. Under the constraint (16), one has

$$v_c \geqslant v^-(T, v)\ \forall\, (T, v) \in \mathscr{S}_a \cap \mathscr{K}. \tag{18}$$

3. GENERALIZED VARIATIONAL PRINCIPLES

Introducing the Lagrange multiplier $v \in \mathscr{U}_a$, the statically admissible constraint in the lower bound theorem can be relaxed by

$$v_s(v, T) = \frac{\int_\Omega T_{ij}\varepsilon_{ij}(v)\,d\Omega - \int_\Omega W^*(T)\,d\Omega - \int_\Omega b_i v_i\,d\Omega}{\int_{\Gamma_t} t_i v_i\,d\Gamma}. \tag{19}$$

Theorem 3. *For any given* $(v, T) \in \mathscr{U}_a \times \mathscr{S}$, *the stationary points* (u, S) *of* $v_s(v, T)$ *solve the boundary value problem* (1). *If the gap function G satisfies*

$$G(v, T) \geqslant 0 \quad \forall\,(v, T) \in \mathscr{U}_a \times \mathscr{S}, \tag{20}$$

then (u, S) *min–maximizes* $v_s(v, T)$, *and*

$$v_c = v_s(u, S) = \inf_{v \in \mathscr{U}_a} \sup_{T \in \mathscr{S}} v_s(v, T). \tag{21}$$

Proof. According to Gao and Strang (1989), the Lagrangian of boundary value problem (1) can be given by

$$L(v, T) = \int_\Omega T_{ij}\varepsilon_{ij}(v)\,d\Omega - \int_\Omega W^*(T)\,d\Omega - \int_\Omega b_i v_i\,d\Omega - \int_{\Gamma_t} v_c t_i v_i\,d\Gamma.$$

It is easy to prove that

$$\delta v_s(u, S; v, T) = 0 \Leftrightarrow \delta L(u, S; v, T) = 0 \quad \forall\,(v, T) \in \mathscr{U}_a \times \mathscr{S}.$$

The stationary condition for Lagrangian L yields Euler–Lagrange equations:

$$[(\delta_{ik} + u_{i,k})S_{kj}]_{,j} - b_i = 0 \quad \text{in } \Omega,$$
$$\varepsilon_{ij}(u) \in \partial W^*(S_{ij}) \quad \text{in } \Omega,$$
$$[(\delta_{ik} + u_{i,k})S_{kj}]n_j - v_c t_i = 0 \quad \text{on } \Gamma_t.$$

It is proved that the stationary points of v_s solve the boundary value problem (1). Moreover, for any given $v \in \mathscr{U}$,

$$\sup_{T \in \mathscr{S}} v_s(v, T) = \frac{\sup_T \left\{\int_\Omega T_{ij}\varepsilon_{ij}(v)\,d\Omega - \int_\Omega W^*(T)\,d\Omega\right\} - \int_\Omega b_i v_i\,d\Omega}{\int_{\Gamma_t} t_i v_i\,d\Gamma}$$
$$= \frac{\int_\Omega W(\varepsilon(v))\,d\Omega - \int_\Omega b_i v_i\,d\Omega}{\int_{\Gamma_t} t_i v_i\,d\Gamma} = v_u(v).$$

Recalling Theorem 1, variational eqn (21) is proved. Q.E.D.

It should be emphasized that the order of inf–sup in saddle point problem (21) can not be exchanged, i.e.

$$\inf_{v} \sup_{T} v_s(v, T) \not\equiv \sup_{T} \inf_{v} v_s(v, T).$$

Over the statically admissible space $\mathscr{S}_a$, v_s is degenerated to

$$v_d(v, T) = v^-(v, T) - \frac{G(v, T) + \int_\Omega W^*(T)\, d\Omega}{\int_{\Gamma_t} t_i v_i\, d\Gamma}. \tag{22}$$

For any given $(T, v) \in \mathscr{S}_a$, if $G(v, T) \geqslant 0$, it is obvious that

$$v_f(T, v) \geqslant v_d(v, T). \tag{23}$$

4. CONSTRUCTION OF THE SUPERPOTENTIAL

The simplest constructions of the superpotential W^* can be given by (see Gao, 1988; Gao and Hwang, 1985):

$$W_p^*(T, \alpha) = \frac{1}{2\alpha} f^2(T)\Phi(f) \quad \text{in}\, \Omega, \tag{24}$$

where $\alpha > 0$ is a penalty factor and Φ is a jump function

$$\Phi(f) = \begin{cases} +1 & \text{if}\, f > 0 \\ 0 & \text{if}\, f \leqslant 0. \end{cases} \tag{25}$$

It is obvious that for any given $T \in \mathscr{S}$,

$$W^*(T) = \sup_{\alpha > 0} W_p^*(T, \alpha) = \lim_{\alpha \to 0^+} W_p^*(T, \alpha). \tag{26}$$

Let

$$v_p(v, T; \alpha) = \frac{\int_\Omega T_{ij}\varepsilon_{ij}(v)\, d\Omega - \int_\Omega W_p^*(T, \alpha)\, d\Omega - \int_\Omega b_i v_i\, d\Omega}{\int_{\Gamma_t} t_i v_i\, d\Gamma}. \tag{27}$$

The penalty-type saddle point principle can be established by substituting eqn (26) into eqn (21), if inequality (20) holds

$$v_c = \lim_{\alpha \to 0^+} \inf_{v \in \mathscr{K}_a} \sup_{T \in \mathscr{S}} v_p(v, T; \alpha). \tag{28}$$

Similarly, by substituting eqn (26) into inequality (17), the penalty-type bound theorems can be obtained as follows:

Theorem 4. *If the solution* (S_α, u_α) *of the penalty optimal problem*

$$v_0(S_\alpha, u_\alpha) = \lim_{\alpha\to 0^+} \sup_{(T,v)\in\mathscr{S}_a} \left\{v^-(T,v) - \int_\Omega W_p^*(T,\alpha)\,d\Omega\right\}, \tag{29}$$

satisfies

$$G(u,S) - G(u_\alpha, S_\alpha) + G(u-u_\alpha, S_\alpha) \geqslant 0,$$

then $v_0(S_\alpha, u_\alpha)$ *gives an optimal lower bound of the plastic limit load multiplier.*

The dual-type construction of the complementary plastic superpotential is given by Gao (1988a):

$$W_d^*(T,\lambda) = \lambda f(T)\Phi(f), \tag{30}$$

where the Lagrange multiplier $\lambda \geqslant 0$ is the dual variable of yield function $f(T)$. One should have

$$W^*(T) = \sup_{\lambda\geqslant 0} W_d^*(T,\lambda) \quad \forall\, T\in\mathscr{S}. \tag{31}$$

Substituting eqn (31) into eqn (19), a dual-type saddle point variational principle can be established under constraint (20).

By combining eqn (24) with eqn (30), one has the penalty–duality constructions of the superpotential:

$$W_{pd}^*(T,v,\lambda;\alpha) = \frac{\alpha}{2}\left\{\left[\lambda + \frac{1}{\alpha} f(T)\right]^2 \Phi(f_\alpha) - \lambda^2\right\}, \tag{32}$$

in which $f_\alpha = \lambda + (1/\alpha)f(T)$ is the so-called penalty–dual dividing domain function (Gao, 1988a).

Theorem 5. *There exists a penalty factor* $\alpha^* > 0$, *such that for any given* $\alpha \in (0, \alpha^*]$, *the optimal plastic limit load multiplier is*

$$v_0(S_\alpha, u_\alpha) = \sup_{(T,v)\in\mathscr{S}_a} \inf_{\lambda\geqslant 0} \left\{v^-(T,v) - \int_\Omega W_{pd}^*(T,\lambda;\alpha)\,d\Omega\right\}. \tag{33}$$

If the gap function G satisfies the constraint

$$G(u,S) - G(u_\alpha, S_\alpha) + G(u-u_\alpha, s_\alpha) \geqslant 0, \tag{34}$$

then $v_0(S_\alpha, u_\alpha)$ *gives an optimal lower bound of the plastic limit load multiplier.*

Proof. It is easy to prove (Gao and Hwang, 1985, 1986) that for any given $\alpha^* > 0$, there exists $\alpha \in (0, \alpha^*]$ such that

$$W^*(T) = \sup_{\lambda\geqslant 0} W_{pd}^*(T,\lambda;\alpha) \quad \forall\, T\in\mathscr{S}.$$

Substituting into eqn (15) and recalling Theorem 2, one has $v_c \geqslant v_0(S_\alpha, u_\alpha)$ if constraint (34) is satisfied.

Based on this theorem, an efficient numerical algorithm for optimal lower bounds of the limit plastic load multiplier can be suggested (see Gao, 1988a,b; Gao and Hwang, 1985).

5. APPLICATION

Consider a simply supported beam, fully restrained from axial motion and subjected to a uniformly distributed load. For the sake of simplicity, the axial component of the

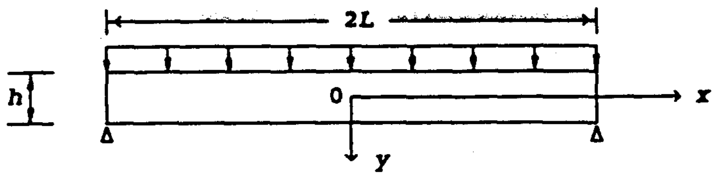

Fig. 1. Beam under a uniform loading.

displacement is neglected. The geometrical parameters involved and the coordinate systems are defined in Fig. 1. The governing equations in this case have the forms:

$$\left.\begin{aligned} M''+(Nw')'+v_c p &= 0, \\ N' &= 0, \\ \varepsilon &= \tfrac{1}{2}w'^2, \\ \kappa &= -w'', \end{aligned}\right\} \tag{35}$$

where w is the displacement in the y direction and M and N denote the bending moment and axial force, respectively. The corresponding generalized strains are (κ, ε). For a rectangular cross-section of the beam, the yield function is described by:

$$f(m,n) = |m|+n^2-1, \tag{36}$$

in which $m = M/M_b$, $n = N/N_b$, $M_b = \sigma_b h^2/4$, $N_b = \sigma_b h$. It was shown by Symonds and Jones (1972) that when $\delta \geqslant h/2$, the stress profile on the yield condition is reduced to the point $M = 0$, $N = N_b$. The gap function (7) in this case has the form:

$$G(w,n) = \int_0^L \tfrac{1}{2}nw'^2\,\mathrm{d}x.$$

It is easy to see that in this problem, $n \geqslant 0$, $G(w,n) \geqslant 0$, i.e. the structure is stable. For given statically admissible fields (w,m,n), eqn (15) becomes

$$v_l(w,m,n) = v^-(w,m,n)-\int_0^L W^*(m,n)\,\mathrm{d}x. \tag{37}$$

Choose the following statically admissible fields:

$$w = w_0\left(1-\frac{x^2}{L^2}\right), \quad m = \eta\left(1-\frac{x^2}{L^2}\right), \quad n = \mu\delta, \tag{38}$$

where η, $\mu \geqslant 0$ are undetermined parameters, $\delta = w_0/h$. Obviously, at initial state ($\delta = 0$), $w(\delta) = 0$. Substituting (38) into (35) and (36) gives

$$v^-(\delta,\eta,\mu) = \eta+4\mu\delta^2, \quad p = \frac{2M_b}{L^2},$$

$$f(\eta,\mu,\delta) = \eta(1-x^2/L^2)+\mu^2\delta^2-1.$$

Using the penalty–dual type construction of W^*, eqn (37) becomes

Table 1. Numerical results for large deformation plastic beam ($h/L = 0.2$)

$\delta = w_0/h$	0.25	0.50	0.75	1.00	1.50	2.00
η	0.781	0.340	0.289	0.100	0.030	0.020
μ	0.468	0.810	0.844	0.948	0.657	0.490
v_0	1.25	1.97	2.82	3.89	5.94	7.93
Membrane solution	1.00	2.00	3.00	4.00	6.00	8.00

$$v_{pd}(\delta, \eta, \mu, \lambda; \alpha) = \eta + 4\mu\delta^2 - \int_0^L W^*_{pd}(\eta, \mu, \delta, \lambda, \alpha)\,\mathrm{d}x, \tag{39}$$

where

$$W^*_{pd}(\eta, \mu, \delta, \lambda, \alpha) = \begin{cases} \dfrac{1}{2\alpha} f^2(\eta, \mu, \delta) + \lambda f(\eta, \mu, \delta) & \text{if } f_\alpha > 0 \\ -\dfrac{\alpha}{2}\lambda^2 & \text{if } f_\alpha \leqslant 0. \end{cases}$$

For a given $\delta > 0$, one has the optimal problem:

$$v_0(\delta) = \max_{\eta>0} \max_{\mu>0} \min_{\lambda\geqslant 0} v_{pd}(\delta, \eta, \mu, \lambda; \alpha) \quad \forall\, \alpha \in (0, \alpha^*). \tag{40}$$

Choosing the primal values $\alpha_0 = 0.1$, $\lambda_0 = 0$, for given precision $\omega = 0.001$, the penalty-duality algorithm (see Gao, 1988b) gives the numerical results shown in Table 1.

For comparison, Table 1 (last row) shows a purely membrane solution obtained from the set of eqns (38) by assuming $M = \kappa = 0$ and $N = N_b$. Using the dimensional quantities defined earlier, this solution is represented by $v_c = 4\delta$. It can be seen in Table 1 that the bending moment starts at $M = M_b$ and rapidly diminishes, reaching practically zero when the deflection equals the beam thickness. At the same time the membrane force steadily increases towards the fully plastic axial force N_b. In this problem, the constraint (34) is

$$\int_0^L \tfrac{1}{2} N_b(1 - \mu_\alpha \delta)\bar{w}'^2\,\mathrm{d}x \geqslant 0 \quad \forall\, \delta \geqslant 0.5,$$

where $\bar{w}$ is the actual deflection of the beam. According to Theorem 5, v_0 gives the optimal lower bound of the plastic limit load multiplier (see Table 1).

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